Homology of coloured posets: A generalisation of Khovanov's cube construction

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We define a homology theory for a certain class of posets equipped with a pre-sheaf of modules. We show that when restricted to Boolean lattices this homology is isomorphic to the homology of the "cube" complex defined by Khovanov.

Introduction

Given a Boolean lattice which has been "coloured" by assigning to each element x a finite dimensional vector space V_x and to each pair $x \leqslant y$ a linear map $V_x \to V_y$, one can construct a chain complex using Khovanov's "cube" construction as found in his celebrated paper on the categorification of the Jones polynomial [Kho00]. The homology of complexes arising in this way plays a central role in link homology theories such as Khovanov homology and Khovanov–Rozansky homology. Recently, Heegaard–Floer knot homology has also been interpreted in terms of the homology of a complex coming from a coloured Boolean lattice. Khovanov's construction relies on specific properties of Boolean lattices and the question that motivates the current paper is: can one define a homology theory for a more general class of "coloured" posets, which for Boolean lattices gives the homology arising from Khovanov's cube complex?

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We provide one possible way of affirmatively answering this question. The labelling of vertices and edges referred to above must satisfy certain compatibility conditions making a "colouring" nothing other than a pre-sheaf of modules over the poset considered as a category (or a representation, according to taste). In this paper we define a chain complex for an arbitrary poset with 1 equipped with such a pre-sheaf. We refer to such posets as *coloured posets*, and they form the objects of a category so that by passing to homology we get a functor to graded modules. This functor satisfactorily answers the above question: for coloured Boolean lattices the result is isomorphic to the homology of Khovanov's cube complex, an outcome not *a priori* obvious.

We begin in Section 1 by studying the category of coloured posets $\mathcal{CP}_R$ over a ring R. We provide a number of examples and several basic constructions. In Section 2 we define a functor $\mathcal{S}_*$ from $\mathcal{CP}_R$ to chain complexes over R which generalises the well-known order homology of a poset to the situation where one has a local system of coefficients. The resulting homology $H_*(P,\mathcal{F})$ is what we refer to as the homology of the coloured poset $(P,\mathcal{F})$. We show that the chain complex $\mathcal{S}_*(P,\mathcal{F})$ is homotopy equivalent to a much smaller complex $\mathcal{C}_*(P,\mathcal{F})$, paralleling the situation in topology where the full simplicial chain complex on a space is cut down by throwing away degeneracies.

The main technical result is in Section 3 where we show that a coloured poset obtained by gluing two coloured posets together by a morphism gives rise to a long exact sequence in homology (see Theorem 21). We give a brief tutorial on Khovanov's cube complex in Section 4, which in the context of this paper is a chain complex $\mathcal{K}_*(\mathbb{B},\mathcal{F})$ associated to a coloured Boolean lattice $(\mathbb{B},\mathcal{F})$. We denote its homology by $KH_*(\mathbb{B},\mathcal{F})$, which is *a priori* very different from the homology $H_*(\mathbb{B},\mathcal{F})$ of the previous paragraph. In Section 5 we present the main result, namely the agreement of the coloured poset homology with the Khovanov's cube homology for coloured Boolean lattices. We construct a chain map ϕ from the cube complex $\mathcal{K}_*(\mathbb{B},\mathcal{F})$ to $\mathcal{C}_*(\mathbb{B},\mathcal{F})$, and our main result, given as Theorem 24 in Section 5, is

Main Theorem. *Let $(\mathbb{B},\mathcal{F})$ be a coloured Boolean lattice. Then $\phi:\mathcal{K}_*(\mathbb{B},\mathcal{F})\to\mathcal{C}_*(\mathbb{B},\mathcal{F})$ is a quasi-isomorphism, yielding isomorphisms,*

$$KH_n(\mathbb{B},\mathcal{F}) \xrightarrow{\cong} H_n(\mathbb{B},\mathcal{F}).$$

In certain cases the theorem can be strengthened. For example (see Section 5, Corollary 28):

Corollary. *If the ground ring R of the colouring $\mathcal{F}:\mathbb{B}\to\mathcal{M}\mathrm{od}_R$ is a field, then $\phi:\mathcal{K}_n(\mathbb{B},\mathcal{F})\to\mathcal{C}_n(\mathbb{B},\mathcal{F})$ is a homotopy equivalence.*

1. Coloured posets

The principal characters in our story, coloured posets, are partially ordered sets (posets) whose elements are labeled by R-modules so that there is a homomorphism between the labels of comparable elements satisfying certain compatibility conditions.

We begin by recalling basic poset terminology, for which we will generally follow [Sta97, Chapter 3]. A *poset* $(P,\leqslant)$ is a set P together with a reflexive, anti-symmetric, transitive binary relation $\leqslant$, and a *map of posets* $f:(P,\leqslant)\to(Q,\leqslant')$ is a set map preserving the respective relations, i.e. $f(x)\leqslant' f(y)$ in Q if $x\leqslant y$ in P. One writes $x<y$ when $x\leqslant y$ and $x\neq y$. If $x<y$ and there is no z with $x<z<y$ then we say that y *covers* x, and write $x\prec y$. The covering relation is illustrated via the *Hasse diagram*: the graph with vertices the elements of P, and an edge joining x to y iff $x\prec y$. We will follow the convention that Hasse diagrams will be presented vertically on the page with y drawn above x whenever $x\prec y$.

An ordered *multi-sequence* is a sequence $x_1\leqslant\cdots\leqslant x_k$, of comparable elements. An ordered *sequence* is a multi-sequence with $x_1<\cdots<x_k$. A sequence is *saturated* when it has the form $x_1\prec\cdots\prec x_k$. This differs from the standard poset terminology (where a multi-sequence is called a multi-chain and a sequence a chain) justified by our giving preference to homological notions, where

the term chain is already taken. There is the obvious notion of a $0 \in P$: an element with $x \geqslant 0$ for all $x \in P$; similarly a 1 is an element with $x \leqslant 1$ for all $x \in P$. A poset P is *graded of rank r* if every saturated sequence, maximal under inclusion of sequences, has the same length r. There is then a unique grading or rank function $\mathrm{rk} : P \to \{0, 1, \dots, r\}$ with $\mathrm{rk}(x) = 0$ if and only if x is minimal, and $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$ whenever $x \prec y$. The rank 1 elements are called the *atoms*.

Sometimes our posets will turn out to be *lattices*: posets for which any x and y have a supremum or least upper bound $x \vee y$ (the join of x and y) and an infimum, or greatest lower bound $x \wedge y$ (the meet of x and y). A lattice is *atomic* if every element can be expressed (not necessarily uniquely) as a join of atoms.

Without explicitly mentioning it, we will often consider a poset as a category whose objects are the elements of the poset, and with a unique morphism $x \to y$ between any two comparable elements $x \leqslant y$. A *pre-sheaf* on (or *representation* of) a poset is a covariant functor to some category of modules (warning: this differs from the convention in sheaf theory where the functor is taken to be contravariant).

Here is a primordial example: the Boolean lattice $\mathbb{B} = \mathbb{B}(X)$ on the set X is a lattice isomorphic to the lattice of subsets of X under inclusion. We will often suppress the isomorphism and identify the elements of $\mathbb{B}$ with subsets of X. If X is finite, then $\mathbb{B}$ is graded with $\mathrm{rk}(S) = |S|$ for $S \subseteq X$, and atomic, with atoms the singletons. If the atoms are given some fixed ordering $a_1, \dots, a_r$, then every $x \in \mathbb{B}$ can be expressed *uniquely* as a join,

$$x = \bigvee a_{i_j} = a_{i_1} \vee a_{i_2} \vee \cdots \vee a_{i_k}, \tag{1}$$

where $i_1 < \cdots < i_k$. For $x = a_{i_1} \vee \cdots \vee a_{i_k}$, one has the covering relation $x \prec y$ if and only if the unique expression for y is $y = (a_{i_1} \vee \cdots \vee a_{i_j}) \vee a_\ell \vee (a_{i_{j+1}} \vee \cdots \vee a_{i_k})$.

With these preliminaries out of the way we now make the principal definition. Fix a unital commutative ring R and let $\mathcal{M}\mathrm{od}_R$ be the category of R-modules.

Definition 1. A *coloured poset* $(P, \mathcal{F})$ consists of

- a poset P having a unique maximal element 1_P, and
- a covariant functor $\mathcal{F} : P \to \mathcal{M}\mathrm{od}_R$.

The functor $\mathcal{F}$ will be referred to as the *colouring*.

A *morphism* of coloured posets $(P_1, \mathcal{F}_1) \to (P_2, \mathcal{F}_2)$ is a pair (f, τ) where

- $f : P_1 \to P_2$ is a map of posets, and
- τ is a collection $\{\tau_x\}_{x \in P_1}$ where $\tau_x : \mathcal{F}_1(x) \to \mathcal{F}_2(f(x))$ is an R-module homomorphism.

This data satisfies the following two conditions

1. $f(x) = 1_{P_2}$ if and only if $x = 1_{P_1}$, and
2. (naturality) for all $x \leqslant y$ in P_1, the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}_1(x) \arrow[r, "\mathcal{F}_1(x \leqslant y)"] \arrow[d, "\tau_x"'] & \mathcal{F}_1(y) \arrow[d, "\tau_y"] \\
\mathcal{F}_2(f(x)) \arrow[r, "\mathcal{F}_2(f(x) \leqslant f(y))"] & \mathcal{F}_2(f(y)).
\end{tikzcd}$$

Coloured posets and morphisms between them form a category denoted $\mathcal{CP}_R$.

Thus the colouring associates to each element of the poset an R-module $\mathcal{F}(x)$ and if $x \leqslant y$ then there is an associated map $\mathcal{F}(x \leqslant y) : \mathcal{F}(x) \to \mathcal{F}(y)$. We will often find it convenient to write $\mathcal{F}_x^y$ instead of $\mathcal{F}(x \leqslant y)$. Also, we usually define the morphisms $\mathcal{F}(x \leqslant y)$ just in the cases where $x \prec y$, as all the others can be recovered from these by repeated composition.

Given a map of (uncoloured) posets $f : P_1 \to P_2$, the composite $\mathcal{F}_2 \circ f : P_1 \to \mathcal{M}\mathrm{od}_R$ defines another colouring on P_1. Condition 2 in the above definition is merely stating that τ is a natural transformation (of functors $P_1 \to \mathcal{M}\mathrm{od}_R$) from $\mathcal{F}_1$ to $\mathcal{F}_2 \circ f$.

The notion of a coloured poset is a rather general one encompassing many interesting examples:

Example 2. Let P be a poset with unique maximal element and A an R-module. The *constant colouring* on P by A is defined by the colouring functor $\mathcal{F} : P \to \mathcal{M}\mathrm{od}_R$ given by $\mathcal{F}(x) = A$ and $\mathcal{F}_x^y = \mathrm{id}_A$ for all $x \leqslant y$.

Example 3. Let X be a topological space and P the poset of open subsets partially ordered by *reverse* inclusion. A colouring is equivalent to a pre-sheaf of R-modules on X.

Example 4. Let G be a group. Then the poset of Abelian p-subgroups for p a prime is naturally a coloured poset, by colouring an element of the poset with the subgroup it corresponds to. Homomorphisms are just the inclusions.

Example 5. The cube construction arising in Khovanov homology [Kho00] is a colouring of a Boolean lattice associated to a link diagram. More recent link homology theories, such as Khovanov–Rozansky homology, are also defined as the homology of the cube complex of a certain coloured Boolean lattice associated to a link diagram [KR08a,KR08b].

Example 6. Although it has more geometric origins, knot Floer homology now has a completely combinatorial description involving a coloured Boolean lattice, along the lines of Khovanov homology [OS04].

Example 7. A colouring of a certain Boolean lattice associated to a graph, first defined by Helme-Guizon and Rong (see [HGR05]), gives a graph homology that is related to the chromatic polynomial.

There are two interesting constructions with coloured posets which we now discuss.

Products. The *product* $(P_1, \mathcal{F}_1) \times (P_2, \mathcal{F}_2) = (P, \mathcal{F})$, has underlying poset P the direct product of the P_i, i.e. the poset with elements $(a, b) \in P_1 \times P_2$ with $(a, b) \leqslant (a', b')$ iff $a \leqslant a'$ and $b \leqslant b'$. The colouring is $\mathcal{F}(a, b) = \mathcal{F}_1(a) \otimes_R \mathcal{F}_2(b)$ and $\mathcal{F}_{(a,b)}^{(a',b')} = (\mathcal{F}_1)_a^{a'} \otimes (\mathcal{F}_2)_b^{b'}$.

For example, if $\mathbb{B}_i$ $(i = 1, 2)$ are Boolean latices of rank r_i (isomorphic to the lattice of subsets of X_i), then $\mathbb{B}_1 \times \mathbb{B}_2$ is Boolean of rank $r_1 + r_2$ (isomorphic to the lattice of subsets of $X_1 \sqcup X_2$). If the $\mathbb{B}_i$ are coloured by $\mathcal{F}_i$, we have a picture like Fig. 1, in the case $r_1 = 1$, $r_2 = 2$, and where we have abbreviated $U_x := \mathcal{F}_1(x)$, $V_x := \mathcal{F}_2(x)$.

Gluing along a morphism. Let $(P_1, \mathcal{F}_1)$ and $(P_2, \mathcal{F}_2)$ be coloured posets and let $(f, \tau) : (P_1, \mathcal{F}_1) \to (P_2, \mathcal{F}_2)$ be a morphism of coloured posets. We can construct a new coloured poset $(P_1, \mathcal{F}_1) \cup_f (P_2, \mathcal{F}_2)$ by "gluing" P_1 to P_2 using the map f.

The underlying set of $(P_1, \mathcal{F}_1) \cup_f (P_2, \mathcal{F}_2)$ is $P_1 \cup P_2$, the union of elements on P_1 and P_2. The partial order on this set is defined as follows.

- If $a, a' \in P_i$ then $a \leqslant a'$ iff $a \leqslant a'$ in P_i;
- If $a \in P_1$ and $a' \in P_2$ then $a \leqslant a'$ iff $f(a) \leqslant a'$ in P_2.

We will denote this poset by $P_1 \cup_f P_2$.

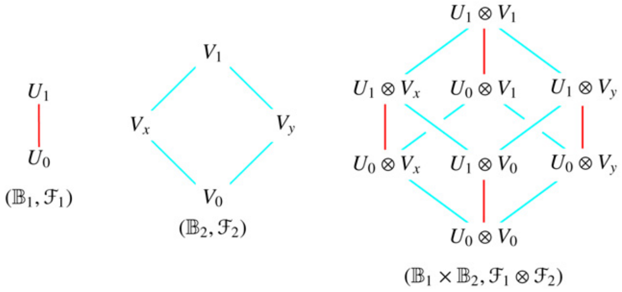


Fig. 1. The product of coloured Boolean lattices of ranks 1 and 2, yielding a coloured Boolean lattice of rank 3.

The colouring functor $\mathcal{F} : P_1 \cup_f P_2 \to \mathcal{M}\mathrm{od}_R$ is defined as follows. For an object $a \in P_i$ set $\mathcal{F}(a) = \mathcal{F}_i(a)$. For a morphism $a \leqslant a'$ we define $\mathcal{F}_a^{a'} : \mathcal{F}(a) \to \mathcal{F}(a')$ as follows.

- If $a, a' \in P_i$ then $\mathcal{F}_a^{a'} = (\mathcal{F}_i)_a^{a'}$, and
- if $a \in P_1$ and $a' \in P_2$ then as part of the morphism (f, τ) there is a map $\tau_a : \mathcal{F}_1(a) \to \mathcal{F}_2(f(a))$. Since $f(a) \leqslant a'$ in P_2 there is a map $(\mathcal{F}_2)_{f(a)}^{a'} : \mathcal{F}_2(f(a)) \to \mathcal{F}_2(a')$. In this case set

$$\mathcal{F}_a^{a'} = (\mathcal{F}_2)_{f(a)}^{a'} \circ \tau_a.$$

Lemma 8. $(P_1 \cup_f P_2, \mathcal{F})$ *is a coloured poset.*

Proof. It is routine to check that $P_1 \cup_f P_2$ is a poset and moreover that 1_{P_2} provides a unique maximal element. To verify that $\mathcal{F}$ is a functor the only real issue is composition. Suppose $a \leqslant a' \leqslant a''$ then we must check $\mathcal{F}_{a'}^{a''} \circ \mathcal{F}_a^{a'} = \mathcal{F}_a^{a''}$. There are a number of cases to consider. If $a, a' \in P_1$ and $a'' \in P_2$, then the identity to check is given by the outside routes around the following diagram.

$$\begin{tikzcd}
\mathcal{F}_1(a) \arrow[r, "(\mathcal{F}_1)_a^{a'}"] \arrow[d, "\tau_a"'] & \mathcal{F}_1(a') \arrow[r, "\tau_{a'}"] & \mathcal{F}_2(f(a')) \arrow[d, "(\mathcal{F}_2)_{f(a')}^{a''}"] \\
\mathcal{F}_2(f(a)) \arrow[rru, "(\mathcal{F}_2)_{f(a)}^{f(a')}"'] \arrow[rr, "(\mathcal{F}_2)_{f(a)}^{a''}"'] & & \mathcal{F}_2(a'')
\end{tikzcd}$$

The right-hand triangle commutes courtesy of the functoriality of $\mathcal{F}_2$ and the left-hand triangle commutes because of the naturality of τ, thus the square commutes. The other cases are simpler and omitted. $\square$

Example 9. Let $(\mathbb{B}_i, \mathcal{F}_i)$ for $i = 0, 1$, be coloured Boolean lattices of the same rank (so both isomorphic to the lattice of subsets of X) and $(f, \tau) : (\mathbb{B}_0, \mathcal{F}_0) \to (\mathbb{B}_1, \mathcal{F}_1)$ a morphisms of coloured posets with f an isomorphism. Then $(\mathbb{B}_0, \mathcal{F}_0) \cup_f (\mathbb{B}_1, \mathcal{F}_1)$ is a Boolean lattice, isomorphic to the lattice of subsets of $X \cup f(0_{\mathbb{B}_0})$; see Fig. 2 for the rank $|X| = 2$ case.

Turning it around, any coloured Boolean lattice $\mathbb{B}$ of rank r can be decomposed $\mathbb{B} = \mathbb{B}_0 \cup_f \mathbb{B}_1$ in a number of ways, each corresponding to a pair of opposite faces of the "cube": with the atoms ordered $a_1, \ldots, a_r$, and a_ℓ a fixed atom, let $\mathbb{B}_0$ be the subposet consisting of $0_{\mathbb{B}}$ and those $x \in \mathbb{B}$ for which the join (1) does not contain a_ℓ, and $\mathbb{B}_1$ those x where it does. Then the $\mathbb{B}_i$ are sub-Boolean of rank $r - 1$.

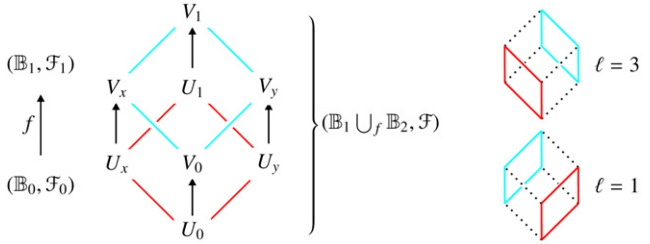


Fig. 2. Gluing coloured Boolean lattices of rank 2 along a morphism to give a coloured Boolean lattice of rank 3 (left), and the three decompositions of a rank 3 lattice as glued rank 2 lattices: for $\ell = 2$ (left) and $\ell = 1, 3$ (right).

For $x \in \mathbb{B}_0$, define $f(x) = x \vee a_\ell$, and $\tau_x := \mathcal{F}_x^{f(x)}$. Fig. 2 illustrates the rank three case, with the three decompositions (clockwise from the main picture) for $\ell = 2, 3$ and 1. The last will play a key role in Sections 3–5.

2. The homology of a coloured poset

Poset homology was pioneered by Folkman and Rota, amongst others. We will make no attempt to summarise this vast area beyond our immediate needs, but to whet the readers appetite we mention a couple of fruitful applications: it provides an organizing principle in group representation theory, where group actions on posets lead to representations on the poset homology [CR87, §66]; in the theory of hyperplane arrangements it plays a key role, where P is the intersection lattice of the arrangement, see, e.g. [OT92, §4.5]. The basic principle is to pass from posets to abstract simplicial complexes. Recall that an abstract simplicial complex with vertex set X is a subset $\Delta \subset 2^X$ such that $\{x\} \in \Delta$ iff $x \in X$, and $\sigma \in \Delta, \tau \subset \sigma \Rightarrow \tau \in \Delta$. The k-simplicies Δ_k are the $k+1$-element subsets and the empty set $\emptyset \in \Delta$ is the unique (-1)-simplex. If P is a poset then the order complex $\Delta(P)$ has $X = P$ and k-simplicies the ordered sequences $\sigma = (x_0 < x_1 < \cdots < x_k)$ of length $k+1$. If P has a 1, then $\Delta(P)$ is a cone on $\Delta(P \setminus 1)$ hence contractible (this is standard, but see for example [OT92, Lemma 4.96]). A similar thing is true if P has a 0, so one normally takes the order complex on the poset with 0's and 1's removed: the so-called Folkman complex. More details on poset topology and homology can be found in [Bjö95]. One can rephrase most statements in traditional poset topology in terms of classifying spaces of categories if one wishes.

Our purpose in this section is to incorporate a colouring of P into this scheme. Essentially this amounts to considering a local coefficient system (given by the colouring) on the order complex. This has already made a brief appearance in the literature on hyperplane arrangements (see [OT92, §4.6]). We use the following notation: an ordered multi-sequence $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ will be abbreviated to $\mathbf{x} = x_1 x_2 \cdots x_n$, and we will write $1 := 1_P$.

If $(P, \mathcal{F})$ is a coloured poset we define the chain complex $\mathcal{S}_*(P, \mathcal{F})$ by setting,

$$\mathcal{S}_k(P, \mathcal{F}) = \bigoplus_{\substack{x_1 x_2 \cdots x_k \\ x_i \in P \setminus 1}} \mathcal{F}(x_1), \tag{2}$$

for $k > 0$. Thus we have one direct summand for each length k multi-sequence $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k$ in $P \setminus 1$. A typical element can thus be written $\sum \lambda_{\mathbf{x}} \cdot \mathbf{x}$, where the sum is over a finite set of length k multi-sequences $\mathbf{x} = x_1 x_2 \cdots x_k$ and $\lambda_{\mathbf{x}} \in \mathcal{F}(x_1)$, and where it is important to remember that the sequence $x_1 x_2 \cdots x_k$ may contain the same element repeated a number of times. For $k = 0$ set

$$\mathcal{S}_0(P, \mathcal{F}) = \mathcal{F}(1),$$

and for $k < 0$, we have $\mathcal{S}_k(P, \mathcal{F}) = 0$. The differential $d_k : \mathcal{S}_k(P, \mathcal{F}) \to \mathcal{S}_{k-1}(P, \mathcal{F})$ is defined for $k > 1$ by

$$d_k(\lambda x_1 x_2 \cdots x_k) = \mathcal{F}_{x_1}^{x_2}(\lambda) x_2 \cdots x_k - \sum_{i=2}^{k} (-1)^i \lambda x_1 \cdots \widehat{x_i} \cdots x_k,$$

and d_1 is defined by

$$d_1(\lambda x) = \mathcal{F}_x^1(\lambda).$$

Lemma 10. *$\mathcal{S}_*(P, \mathcal{F})$ is a chain complex.*

Proof. We need to show $d^2 = 0$. It is not hard to see that $d_{k-1}(d_k(\lambda x_1 x_2 \cdots x_k))$ is a sum of terms of the form $\mu x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_k$, where each indexing multi-sequence appears exactly twice. We need to check that such pairs have opposite signs. One such term arises by the deletion of x_i and then x_j, with its sign being $(-(-1)^i) \times (-(-1)^{j-1}) = (-1)^{i+j-1}$. On the other hand if x_j is deleted first then the sign is $(-(-1)^j) \times (-(-1)^i) = (-1)^{i+j}$, hence the pairs cancel. $\square$

Given a morphism of coloured posets $(f, \tau) : (P_1, \mathcal{F}_1) \to (P_2, \mathcal{F}_2)$ there is an induced map

$$f_* : \mathcal{S}_*(P_1, \mathcal{F}_1) \to \mathcal{S}_*(P_2, \mathcal{F}_2)$$

defined by

$$\lambda x_1 x_2 \cdots x_k \mapsto \tau_{x_1}(\lambda) f(x_1) f(x_2) \cdots f(x_k).$$

Lemma 11. *f_* is a well-defined chain map.*

Proof. Clearly $\tau_{x_1}(\lambda) f(x_1) f(x_2) \cdots f(x_k)$ is an element of $\mathcal{S}_*(P_2, \mathcal{F}_2)$ and by the first condition for a morphism of coloured posets we also have $f(x_k) \neq 1_{P_2}$. To see that f_* is a chain map we calculate

$$d\big(f_*(\lambda x_1 x_2 \cdots x_k)\big) = \mathcal{F}_{f(x_1)}^{f(x_2)}\big(\tau_{x_1}(\lambda)\big) f(x_2) \cdots f(x_k) + \Phi,$$

and

$$f_*\big(d(\lambda x_1 x_2 \cdots x_k)\big) = \tau_{x_2}\big(\mathcal{F}_{x_1}^{x_2}(\lambda)\big) f(x_2) \cdots f(x_k) + \Phi,$$

for $\Phi = -\sum_{i=2}^{k} (-1)^i \tau_{x_1}(\lambda) f(x_1) \cdots \widehat{f(x_i)} \cdots f(x_k)$. These are equal by the naturality of τ. $\square$

We have thus defined a covariant functor

$$\mathcal{S}_* : \mathcal{CP}_R \to \mathrm{Ch}_R,$$

from coloured posets to chain complexes over R. Finally, we define the *homology of the coloured poset* $(P, \mathcal{F})$ to be

$$H_n(P, \mathcal{F}) = H_n\big(\mathcal{S}_*(P, \mathcal{F})\big).$$

Since homology is a functor from chain complexes to graded R-modules we therefore have a covariant functor

$$H_* : \mathcal{CP}_R \to Gr\,\mathcal{M}\mathrm{od}_R.$$

Just as for homology of spaces we can cut down the size of the chain complex by factoring out redundancies. We define $\mathcal{C}_*(P,\mathcal{F})$ identically to $\mathcal{S}_*(P,\mathcal{F})$, but with the additional requirement that the $\mathbf{x} = x_1x_2\cdots x_k$ appearing in (2) are now *sequences* $\mathbf{x} = x_1 < x_2 < \cdots < x_k$. More precisely, for $k > 0$ let

$$\mathcal{C}_k(P,\mathcal{F}) = \bigoplus_{\substack{x_1x_2\cdots x_k \\ x_i < x_{i+1}}} \mathcal{F}(x_1),$$

with the $x_i \in P \setminus 1$ as before. Thus we have a direct summand for each length k ordered sequence $x_1 < x_2 < \cdots < x_k$ in $P \setminus 1$. For $k = 0$ we set

$$\mathcal{C}_0(P,\mathcal{F}) = \mathcal{F}(1),$$

and $\mathcal{C}_k(P,\mathcal{F}) = 0$ for $k < 0$. Clearly $\mathcal{C}_k \subset \mathcal{S}_k$, and we define the differential to be the restriction to $\mathcal{C}_k$ of the differential on $\mathcal{S}_k$ (and so $\mathcal{C}_*$ is a subcomplex of $\mathcal{S}_*$). Note that if there is a maximal length r_0 of an ordered sequence in P, then we have $\mathcal{C}_k(P,\mathcal{F}) = 0$ for $k > r_0$, which is not the case for $\mathcal{S}_*$. Nevertheless, it turns out that $\mathcal{C}_*(P,\mathcal{F})$ is homotopy equivalent to $\mathcal{S}_*(P,\mathcal{F})$ as we will now see.

Let $\mathcal{D}_k \subset \mathcal{S}_k$ be the sub-module containing those summands indexed by sequences with at least one repeat i.e. generated by elements of the form $\lambda x_1x_2\cdots x_k$ where $x_i = x_{i+1}$ for at least one i. If $\lambda x_1\cdots xx\cdots x_k$ is one such, then there are only two terms in $d(\lambda x_1\cdots xx\cdots x_k)$ without the repeated x's, and these have opposite signs, hence cancel (if $\lambda xxx_3\cdots x_k$ is the term, then recall that $\mathcal{F}_x^x = \mathrm{id}$). Thus d is closed on $\mathcal{D}_*$ so $\mathcal{D}_* \subset \mathcal{S}_*$ is a subcomplex, and we have proved,

Lemma 12. *There is a decomposition of complexes*

$$\mathcal{S}_*(P,\mathcal{F}) = \mathcal{C}_*(P,\mathcal{F}) \oplus \mathcal{D}_*(P,\mathcal{F}).$$

But the complex $\mathcal{D}_*$ proves not to be interesting, as the following proposition shows. The result is similar to standard results is algebraic topology, but it is simple enough to write down an explicit proof, so we include it for completeness.

Proposition 13. *There is a homotopy equivalence of chain complexes*

$$\mathcal{D}_*(P,\mathcal{F}) \simeq 0 := \left(\ \cdots \xrightarrow{\ 0\ } 0 \xrightarrow{\ 0\ } 0 \xrightarrow{\ 0\ } \cdots\ \right).$$

Proof. It suffices to show that the identity map on $\mathcal{D}_* = \mathcal{D}_*(P,\mathcal{F})$ is null homotopic. For this we need a family of maps $h_i : \mathcal{D}_i \to \mathcal{D}_{i+1}$ such that

$$\mathrm{id} = h_{i-1}d_i + d_{i+1}h_i. \tag{3}$$

Given a multi-sequence $\mathbf{x} = x_1x_2\cdots x_k$ in $\mathcal{D}_*$ we define $p = p(\mathbf{x}) = \min\{i \mid x_i = x_{i+1}\}$, so $p(\mathbf{x})$ is the position of the first repeating element, and let $n = n(\mathbf{x})$ be the number of times x_p repeats. Thus we can write a multi-sequence $\mathbf{x} = x_1x_2\cdots x_k$ as $x_1\cdots x_{p-1}x_p^n x_{p+n}\cdots x_k$, where $x_j \neq x_{j+1}$ for $1 \leqslant j \leqslant p$ and $n \geqslant 2$. Define $h_i : \mathcal{D}_i \to \mathcal{D}_{i+1}$ by

$$h_i\big(\lambda x_1\cdots x_{p-1}x_p^n x_{p+n}\cdots x_k\big) = \begin{cases} (-1)^{p+1}\lambda x_1\cdots x_{p-1}x_p^{n+1}x_{p+n}\cdots x_k & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

To show (3) we consider separately the two cases n odd and even.

The case: n odd.

We have

$$d_i(\lambda\mathbf{x}) = \mathcal{F}_{x_1}^{x_2}(\lambda)x_2\cdots x_{p-1}x_p^n x_{p+n}\cdots x_k - \sum_{j=2}^{p-1}(-1)^j\lambda x_1\cdots\widehat{x}_j\cdots x_{p-1}x_p^n x_{p+n}\cdots x_k$$

$$-(-1)^p\lambda x_1\cdots x_{p-1}x_p^{n-1}x_{p+n}\cdots x_k - \sum_{j=p+n}^{k}(-1)^j\lambda x_1\cdots x_{p-1}x_p^n x_{p+n}\cdots\widehat{x}_j\cdots x_k.$$

For the case $p=1$ the formula looks slightly different, but the reader will have no trouble making the appropriate adjustments. Note how the n terms indexed by $x_1\cdots x_{p-1}x_p^{n-1}x_{p+n}\cdots x_k$ cancel to give a single term when n is odd. Applying h_{i-1} then gives zero on all terms except $-(-1)^p\lambda x_1\cdots x_{p-1}x_p^{n-1}x_{p+n}\cdots x_k$, so that,

$$\begin{aligned} h_{i-1}d_i(\lambda\mathbf{x}) &= h_{i-1}\big(-(-1)^p\lambda x_1\cdots x_{p-1}x_p^{n-1}x_{p+n}\cdots x_k\big)\\ &= -(-1)^p(-1)^{p+1}\lambda x_1\cdots x_{p-1}x_p^n x_{p+n}\cdots x_k = \lambda\mathbf{x}, \end{aligned}$$

resulting in $h_{i-1}d_i(\lambda\mathbf{x}) + d_{i+1}h_i(\lambda\mathbf{x}) = \lambda\mathbf{x} + d_{i+1}(0) = \lambda\mathbf{x}$.

The case: n even.

We compute,

$$\begin{aligned} h_{i-1}d_i(\lambda\mathbf{x}) = {} & (-1)^p\mathcal{F}_{x_1}^{x_2}(\lambda)x_2\cdots x_{p-1}x_p^{n+1}x_{p+n}\cdots x_k\\ & -(-1)^p\sum_{j=2}^{p-1}(-1)^j\lambda x_1\cdots\widehat{x}_j\cdots x_{p-1}x_p^{n+1}x_{p+n}\cdots x_k\\ & -0-(-1)^{p+1}\sum_{j=p+n}^{k}(-1)^j\lambda x_1\cdots x_{p-1}x_p^{n+1}x_{p+n}\cdots\widehat{x}_j\cdots x_k. \end{aligned}$$

We also have $d_{i+1}h_i(\lambda\mathbf{x}) = d_{i+1}((-1)^{p+1}\lambda x_1\cdots x_{p-1}x_p^{n+1}x_{p+n}\cdots x_k)$, which in turn equals

$$\begin{aligned} & (-1)^{p+1}\mathcal{F}_{x_1}^{x_2}(\lambda)x_2\cdots x_{p-1}x_p^{n+1}x_{p+n}\cdots x_k\\ & -(-1)^{p+1}\sum_{j=2}^{p-1}(-1)^j\lambda x_1\cdots\widehat{x}_j\cdots x_{p-1}x_p^{n+1}x_{p+n}\cdots x_k\\ & -(-1)^{p+1}(-1)^p\lambda x_1\cdots x_{p-1}x_p^n x_{p+n}\cdots x_k\\ & -(-1)^{p+1}\sum_{j=p+n}^{k}(-1)^{j+1}\lambda x_1\cdots x_{p-1}x_p^{n+1}x_{p+n}\cdots\widehat{x}_j\cdots x_k. \end{aligned}$$

Thus

$$h_{i-1}d_i(\lambda\mathbf{x}) + d_{i+1}h_i(\lambda\mathbf{x}) = -(-1)^{p+1}(-1)^p\lambda x_1\cdots x_{p-1}x_p^n x_{p+n}\cdots x_k = \lambda\mathbf{x},$$

as required. □

Corollary 14. *There is a homotopy equivalence of chain complexes* $\mathcal{C}_*(P, \mathcal{F}) \simeq \mathcal{S}_*(P, \mathcal{F})$.

In particular, $H_n(P, \mathcal{F}) \cong H_n(\mathcal{C}_*(P, \mathcal{F}))$, a form more amenable to calculation.

We now briefly elaborate on the connection with traditional (uncoloured) poset homology and in particular the assertion that we have a local coefficient system on the order complex. An abstract simplicial complex may be viewed as a category with objects Δ and a unique morphism $\sigma \to \tau$ whenever $\tau \subset \sigma$. A system of local coefficients on Δ is a (covariant) functor $\Delta \to \mathcal{M}\mathrm{od}_R$ (cf. [GM99, §2.4]). One can form the chain complex $\mathcal{B}_*(\Delta, \mathcal{F})$ with

$$\mathcal{B}_k = \bigoplus_{\sigma \in \Delta_k} \mathcal{F}(\sigma),$$

the direct sum over the k-simplicies. If $\sigma = (x_0 < \cdots < x_k)$ is one such and $\sigma_j = (x_0 < \cdots < \widehat{x}_j < \cdots < x_k)$, then the differential is

$$d(\lambda\sigma) = \sum_{j=0}^{k} (-1)^j \mathcal{F}(\sigma \to \sigma_j)(\lambda)\sigma_j.$$

If $\mathcal{F}$ is a constant system of local coefficients, $\mathcal{F}(\sigma) = A$ for all $\sigma \in \Delta$ and some $A \in \mathcal{M}\mathrm{od}_R$, and $\mathcal{F}(\sigma \to \sigma_j) = \mathrm{id}_A$, then this complex is the one appearing in traditional poset topology: if Δ is the Folkman complex of P (i.e. the order complex of $P \setminus 0, 1$) then its homology is the *order homology of* P with coefficients in A.

There is an augmented version $\widetilde{\mathcal{B}}_*(\Delta, \mathcal{F})$ with $\widetilde{\mathcal{B}}_k = \mathcal{B}_k$ for $k \geqslant 0$, and $\widetilde{\mathcal{B}}_{-1} = \mathcal{F}(\emptyset)$. The extended differential $d : \widetilde{\mathcal{B}}_0 \to \widetilde{\mathcal{B}}_{-1}$ is given by the augmentation $d(\lambda\sigma) = \mathcal{F}(\sigma \to \emptyset)(\lambda)$.

Now if P is a poset with 1 and $\mathcal{F} : P \to \mathcal{M}\mathrm{od}_R$ a colouring, then we get a system of local coefficients on the order complex $\mathcal{F}_P : \Delta(P) \to \mathcal{M}\mathrm{od}_R$ given by $\mathcal{F}_P(\sigma) = \mathcal{F}(x_0)$ when $\sigma = (x_0 < \cdots < x_k)$, and $\mathcal{F}_P(\emptyset) = \mathcal{F}(1)$. If $\sigma \to \sigma_j$ is a morphism in $\Delta(P)$ where $\sigma_j = (x_0 < \cdots < \widehat{x}_j < \cdots < x_k)$, then $\mathcal{F}_P(\sigma \to \sigma_j) = \mathcal{F}_{x_0}^{x_1}$ when $j = 0$, and is the identity otherwise. We may restrict $\mathcal{F}_P$ to a system of local coefficients on the subcomplex $\Delta(P \setminus 1)$, and in doing so we get the explicit connection we are looking for,

Proposition 15. $\mathcal{C}_*(P, \mathcal{F}) = \widetilde{\mathcal{B}}_{*-1}(\Delta(P \setminus 1), \mathcal{F}_P)$.

The proof is just of matter of unraveling the various definitions. The advantage of this formulation is that to a certain extent it allows us to appeal to the existing theory of lattice homology. For example, if P is a poset with 0, then the order complex $\Delta(P)$ is a cone on $\Delta(P \setminus 0)$, and so the (reduced) order homology of an uncoloured poset with 0 is trivial. Indeed, the same happens in the coloured case when the colouring is constant:

Example 16. Let P be a poset with minimal element 0_P and $\mathcal{F}$ the constant colouring by the R-module A. Then $\mathcal{C}_*(P, \mathcal{F})$ is acyclic i.e. $H_n(P, \mathcal{F}) = 0$ for all n. This follows immediately from the above and Proposition 15 together with a little care in degrees zero and one.

Example 17. If $(P_1, \mathcal{F}_1)$ and $(P_2, \mathcal{F}_2)$ are coloured posets then there is a decomposition of complexes

$$\mathcal{S}_*(P_1 \cup P_2, \mathcal{F}_1 \cup \mathcal{F}_2) \cong \mathcal{S}_*(P_1, \mathcal{F}_1) \oplus \mathcal{S}_*(P_2, \mathcal{F}_2),$$

inducing an isomorphism,

$$H_*\big((P_1, \mathcal{F}_1) \cup (P_2, \mathcal{F}_2)\big) \xrightarrow{\cong} H_*(P_1, \mathcal{F}_1) \oplus H_*(P_2, \mathcal{F}_2).$$

The essential point here is that elements of P_1 and elements of P_2 are incomparable in $P_1 \cup P_2$ so an ordered sequence $\mathbf{x}$ in $(P_1 \cup P_2) \setminus 1$ is either completely in $P_1 \setminus 1$ or completely in $P_2 \setminus 1$. Moreover the differential respects this splitting.

Cohomology. By defining

$$\mathcal{S}^*(P, \mathcal{F}) = \mathrm{Hom}_R\big(\mathcal{S}_*(P, \mathcal{F}), R\big)$$

one can define *cohomology* $H^*(P, \mathcal{F})$ as the homology of the resulting cochain complex. If R is a field then the universal coefficient theorem gives an isomorphism $H^*(P, \mathcal{F}) \cong H_*(P, \mathcal{F})$.

Example 18. Let P be finite with both a 0 and a 1. Let P^{op} be the opposite poset defined by $x \leqslant y$ in P^{op} if and only if $y \leqslant x$ in P. If we consider P as a category, P^{op} is simply the opposite category. Since P has a 0, P^{op} has a 1. If we have a colouring functor $\mathcal{F} : P \to \mathcal{M}\mathrm{od}_R$ then by composing this with the functor $(-)^\vee : \mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_R$ taking a module A to its dual $\mathrm{Hom}_R(A, R)$, we get a contravariant functor $P \to \mathcal{M}\mathrm{od}_R$. Equivalently, we have a covariant functor $P^{\mathrm{op}} \to \mathcal{M}\mathrm{od}_R$, or in other words a colouring $\mathcal{F}^\vee$ of P^{op}. Explicitly, $\mathcal{F}^\vee(x) = \mathcal{F}(x)^\vee = \mathrm{Hom}_R(\mathcal{F}(x), R)$ and for $g \in \mathrm{Hom}_R(\mathcal{F}(x), R)$ we have $\mathcal{F}^\vee(x < y)(g) = g \circ \mathcal{F}(y < x)$.

In this situation we have the following duality result

$$H_k\big(P^{\mathrm{op}}, \mathcal{F}^\vee\big) \cong H^{n-k}(P, \mathcal{F}),$$

seen by observing that

$$\begin{aligned} \mathcal{S}^{n-k}(P, \mathcal{F}) &= \mathrm{Hom}_R\big(\mathcal{S}_{n-k}(P, \mathcal{F}), R\big) = \mathrm{Hom}_R\Big(\bigoplus_{\mathbf{x}} \mathcal{F}(x_1), R\Big) \\ &= \bigoplus_{\mathbf{x}} \mathrm{Hom}_R\big(\mathcal{F}(x_1), R\big) = \bigoplus_{\mathbf{x}} \mathcal{F}(x_1)^\vee = \mathcal{S}_k\big(P^{\mathrm{op}}, \mathcal{F}^\vee\big). \end{aligned}$$

3. A long exact sequence for the poset obtained by gluing along a morphism

In this section we show that a map $(f, \tau) : (P_1, \mathcal{F}_1) \to (P_2, \mathcal{F}_2)$ of coloured posets yields a long exact sequence in homology for the coloured poset $P_1 \cup_f P_2$ obtained by gluing along the morphism. The main spin-off occurs when we focus on Boolean lattices, where the decomposition of Example 9 yields a long exact sequence in the homology of the three ingredients. This is the main technical tool needed to show that for a Boolean lattice, the coloured poset homology defined in the last section agrees with Khovanov's cube homology which we discuss in Section 4.

Given coloured posets $(P_1, \mathcal{F}_1)$ and $(P_2, \mathcal{F}_2)$ and a morphism $(f, \tau) : (P_1, \mathcal{F}_1) \to (P_2, \mathcal{F}_2)$ we can form the three complexes $\mathcal{C}_*(P_i, \mathcal{F}_i)$ for $i = 0, 1$ and $\mathcal{C}_*(P_1 \cup_f P_2, \mathcal{F})$. It is clear that $\mathcal{C}_*(P_2, \mathcal{F}_2)$ is a sub-module of $\mathcal{C}_*(P_1 \cup_f P_2, \mathcal{F})$, but also, one easily checks that $d(\mathcal{C}_*(P_2, \mathcal{F}_2)) \subset \mathcal{C}_*(P_2, \mathcal{F}_2)$, and so there is a short exact sequence of complexes

$$0 \longrightarrow \mathcal{C}_*(P_2, \mathcal{F}_2) \xrightarrow{\ i\ } \mathcal{C}_*(P_1 \cup_f P_2, \mathcal{F}) \xrightarrow{\ q\ } Q_* \longrightarrow 0,$$

where Q_* is the quotient. This yields a long exact sequence in homology,

$$\cdots \xrightarrow{\ \delta\ } H_n(P_2, \mathcal{F}_2) \xrightarrow{\ i_*\ } H_n(P_1 \cup_f P_2, \mathcal{F}) \xrightarrow{\ q_*\ } H_n(Q_*) \xrightarrow{\ \delta\ } H_{n-1}(P_2, \mathcal{F}_2) \xrightarrow{\ i_*\ } \cdots \tag{4}$$

The n-chain module Q_n of the quotient complex is isomorphic to

$$\bigoplus_{\mathbf{x}} \mathcal{F}(x_1),$$

the direct sum over those sequences $\mathbf{x}$ in P not entirely contained in P_2, i.e. $\mathbf{x} = x_1 \cdots x_n$ with $x_i \in P_1$ or $\mathbf{x} = x_1 \cdots x_j y_1 \cdots y_{n-j}$ where $0 < j < n$ and the $x_i \in P_1$, $y_i \in P_2 \setminus 1$. We will write $\mathbf{x} = x_1 \cdots x_j y_1 \cdots y_{n-j}$ for the generic sequence, with the understanding that $\mathbf{x} = x_1 \cdots x_n$ when $j = n$. The differential is given by $d(\lambda \mathbf{x}) = \alpha_j + \beta_j$, where $\alpha_1 = 0$ and

$$\alpha_j = \mathcal{F}_{x_1}^{x_2}(\lambda) x_2 \cdots x_j y_1 \cdots y_{n-j} + \sum_{k=2}^{j} (-1)^{k-1} \lambda x_1 \cdots \widehat{x}_k \cdots x_j y_1 \cdots y_{n-j},$$

for $j > 1$, and $\beta_n = 0$, and

$$\beta_j = \sum_{k=1}^{n-j} (-1)^{j+k-1} \lambda x_1 \cdots x_j y_1 \cdots \widehat{y}_k \cdots y_{n-j},$$

for $j < n$. We now define $\pi_n : Q_n \to \mathcal{C}_{n-1}(P_1, \mathcal{F}_1)$ by

$$\pi(\lambda x_1 \cdots x_j y_1 \cdots y_{n-j}) = \begin{cases} \lambda x_1 \cdots x_{n-1}, & \text{if } j = n \text{ and } x_n = 1_{P_1}, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

It is routine to check that,

Lemma 19. $\pi : Q_* \to \mathcal{C}_{*-1}(P_1, \mathcal{F}_1)$ *is a chain map.*

In fact while Q_* is *a priori* a great deal bigger than $\mathcal{C}_*(P_1, \mathcal{F}_1)$, it turns out to contain a number of acyclic subcomplexes, allowing us to establish an isomorphism $H_n(Q_*) \cong H_{n-1}(P_1, \mathcal{F}_1)$.

Proposition 20. *The induced map* $\pi_* : H_n(Q_*) \to H_{n-1}(P_1, \mathcal{F}_1)$ *is an isomorphism.*

Delaying the proof of this momentarily, we now combine this proposition with the long exact sequence (4) to obtain the main result of this section.

Theorem 21. *Let* $(P_i, \mathcal{F}_i)$, $i = 1, 2$, *be coloured posets and* $(f, \tau) : (P_1, \mathcal{F}_1) \to (P_2, \mathcal{F}_2)$ *a morphism of coloured posets. Then there is a long exact sequence,*

$$\cdots \longrightarrow H_n(P_2, \mathcal{F}_2) \xrightarrow{i_*} H_n(P_1 \cup_f P_2, \mathcal{F}) \xrightarrow{(\pi q)_*} H_{n-1}(P_1, \mathcal{F}_1) \xrightarrow{\delta \pi_*^{-1}} H_{n-1}(P_2, \mathcal{F}_2) \longrightarrow \cdots$$

where $(P_1 \cup_f P_2, \mathcal{F})$ *is the coloured poset obtained by gluing along the morphism.*

Before proving Proposition 20 we introduce a number of auxiliary complexes that play a role in the analysis of the homology of Q_*. For $p > 0$, fix $\mathbf{x} = x_1 \cdots x_p$ in P_1, and define a complex $A_*^{\mathbf{x}}$ by setting

$$A_q^{\mathbf{x}} = \bigoplus_{\mathbf{x} y_1 \cdots y_q} \mathcal{F}(x_1),$$

i.e. the direct sum over those $x_1 \cdots x_p y_1 \cdots y_q$ where the x's are fixed and the y's in P_2 are allowed to vary. Let $A_0^{\mathbf{x}} = \mathcal{F}(x_1)$. The differential is given by

$$d(\lambda \mathbf{x} y_1 \cdots y_q) = \sum_{k=1}^{q} (-1)^{p+k-1} \lambda \mathbf{x} y_1 \cdots \widehat{y}_k \cdots y_q,$$

and $d(\lambda \mathbf{x} y) \mapsto \lambda \in A_0^{\mathbf{x}}$.

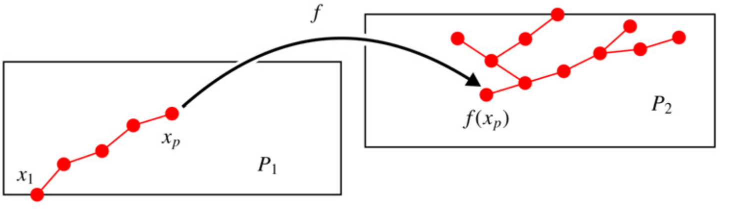
Let $x \in P_1$ and P^x those $y \in P_2$ with $x < y$ in P. Then is easy to see that P^x is a subposet of P with unique minimal element $f(x)$. Let $\varepsilon_p = 1$ when p is even and $\varepsilon_p = (-1)^{p+q}$ when p is odd.

Lemma 22. *The map $\lambda \mathbf{x} y_1 \cdots y_q \mapsto \varepsilon_p \lambda y_1 \cdots y_q$ is an isomorphism of complexes*

$$A_*^{\mathbf{x}} \xrightarrow{\cong} \mathcal{C}_*\big(P^{x_p}, \mathcal{F}_{\mathbf{x}}\big),$$

where $\mathbf{x} = x_1 \cdots x_p$ and $\mathcal{F}_{\mathbf{x}}$ is the constant colouring $\mathcal{F}_{\mathbf{x}}(x) = \mathcal{F}(x_1)$. In particular, $A_*^{\mathbf{x}}$ is acyclic.

The proof is elementary, and no doubt the reader can provide the details by scrutinizing the picture,



while keeping a close eye on the signage. Acyclic-ness follows from Example 16.

We now return to the proof of Proposition 20:

Proof of Proposition 20. Let $A_n \subset Q_n$ be the direct sum $\bigoplus_{\mathbf{x}} \mathcal{F}(x_1)$ over those $\mathbf{x} = x_1 \cdots x_{n-1} x_n$ where $x_n = 1_1 := 1_{P_1}$, the unique maximal element in P_1, and decompose (as modules),

$$Q_n = A_n \oplus D_n.$$

Thus, D_n is the direct sum over those $\mathbf{x}$ in Q_n not finishing at 1_1. It is readily checked that $d(D_n) \subset D_{n-1}$, giving that (D_*, d) is a subcomplex of (Q_*, d).

We now show that D_* is acyclic i.e. $H_n(D_*) = 0$ for all n. We show this by filtering D_* and analyzing the associated spectral sequence. Let $p > 0$, and set

$$F_p D_n = \bigoplus_{\mathbf{x}} \mathcal{F}(x_1),$$

the direct sum over those $\mathbf{x} = x_1 \cdots x_j y_1 \cdots y_{n-j}$ where now $1 \leqslant j \leqslant p$ (and as usual, the $x_i \in P_1$ and the $y_i \in P_2 \setminus 1$). Set $F_p D_* = 0$ for $p < 1$. Thus $F_p D_n$ consists of those modules indexed by sequences of length n which have exited $P_1 \subset P$ after the pth element.

It is easy to check that $F_p D_*$ is a subcomplex of D_*, yielding for each n a bounded filtration,

$$0 = F_0 D_n \subset F_1 D_n \subset \cdots \subset F_{p-1} D_n \subset F_p D_n \subset F_{p+1} D_n \subset \cdots \subset F_n D_n = D_n.$$

This gives rise to a first quadrant spectral sequence converging to H_*D. The E^0-page is

$$E^0_{p,q} = \frac{F_p D_{p+q}}{F_{p-1} D_{p+q}} = \bigoplus_{x_1\cdots x_p y_1\cdots y_q} \mathcal{F}(x_1),$$

with differential $d^0 : E^0_{p,q} \to E^0_{p,q-1}$ defined by

$$d^0(\lambda x_1 \cdots x_p y_1 \cdots y_q) = \sum_{k=1}^{q} (-1)^{p+k-1} \lambda x_1 \cdots x_p y_1 \cdots \widehat{y}_k \cdots y_q.$$

Note that $E^0_{0,q} = 0$ for all q, and thus $E^\infty_{0,q} = 0$. To compute the E^1-page we fix p and consider separately the cases $q > 0$ and $q = 0$.

The case $q > 0$.

We show that any cycle in $E^0_{p,q}$ is also a boundary, and thus $E^1_{p,q} = 0$. Let

$$\sigma = \sum_i \lambda_i x_{i,1} \cdots x_{i,p} y_{i,1} \cdots y_{i,q}$$

be a general element of $E^0_{p,q}$. Let $\mathbf{x} = x_1 \cdots x_p$ be a *fixed* sequence in P_1 and

$$\sigma^{\mathbf{x}} = \sum_j \lambda \mathbf{x} y_{j,1} \cdots y_{j,q}$$

the sum of those terms in σ with $x_{i,1} \cdots x_{i,p} = \mathbf{x}$. Then $\sigma = \sum_{\mathbf{x}} \sigma^{\mathbf{x}}$, the sum over those $\mathbf{x}$ appearing as initial segments in σ, and

$$d^0 \sigma = \sum_{\mathbf{x}} d^0 \sigma^{\mathbf{x}}.$$

Thus, if $A^{\mathbf{x}}_*$ is the complex defined immediately prior to Lemma 22, now a subcomplex of $E^0_{p,*}$, then $\sigma^{\mathbf{x}} \in A^{\mathbf{x}}_* \subset E^0_{p,*}$, and $d^0 \sigma^{\mathbf{x}} \in A^{\mathbf{x}}_{*-1} \subset E^0_{p,*-1}$. Also, if $\mathbf{x} \neq \mathbf{w}$ then $A^{\mathbf{x}}_* \cap A^{\mathbf{w}}_* = \{0\}$ as subcomplexes of $E^0_{p,*}$, and so σ is a cycle if and only if each $\sigma^{\mathbf{x}}$ is a cycle. But the $A^{\mathbf{x}}_*$ are acyclic, so $\sigma^{\mathbf{x}} = d^0 \tau^{\mathbf{x}}$ for some $\tau^{\mathbf{x}} \in A^{\mathbf{x}}_{*+1} \subset E^0_{p,*+1}$, giving $\sigma = d^0(\sum \tau^{\mathbf{x}})$, and thus $E^1_{p,q} = 0$ as claimed.

The case $q = 0$.

Here $d^0 = 0$ and so the cycles are all of $E^0_{p,0} = \bigoplus_{\mathbf{x}} \mathcal{F}(x_1)$, where $\mathbf{x} = x_1 \cdots x_p$ with $x_p \neq 1_1$. We show that $d^0 : E^0_{p,1} \to E^0_{p,0}$ is onto and conclude that $E^1_{p,0} = 0$. If $\lambda x_1 \cdots x_p$ is an element with $x_p \neq 1_1$, then $f(x_p) \in P_2$ is $\neq 1_2 = 1$, and $x_p < f(x_p)$. We then have $d^0(\lambda x_1 \cdots x_p f(x_p)) = \lambda x_1 \cdots x_p$ as required.

Thus the E^1-page of the spectral sequence is entirely trivial, so that in the induced filtration of H_*D,

$$\cdots \subset F_{p-1} H_n(D_*) \subset F_p H_n(D_*) \subset F_{p+1} H_n(D_*) \subset \cdots \subset H_n(D_*),$$

we have trivial quotients. Thus $F_{p-1} H_n = F_p H_n$ for all p and n. As $F_0 H_n D = 0$, we conclude that $H_n(D_*) = 0$ as claimed.

To finish the proof observe that there is a short exact sequence

$$0 \longrightarrow D_* \longrightarrow Q_* \xrightarrow{\pi} A_* \longrightarrow 0,$$

whose associated homology long exact sequence, together with the acyclic-ness of D_*, gives that the quotient map $\pi : Q_* \to A_*$ induces isomorphisms $\pi_* : H_n(Q_*) \to H_n(A_*)$. Now, $A_n = \bigoplus_{x_1 \cdots x_n} \mathcal{F}(x_1)$ with $x_n = 1_1$, and thus the complex A_* can be identified with $\mathcal{C}_{*-1}(P_1, \mathcal{F}_1)$. Under this identification the map π above is the map $\pi : Q_* \to \mathcal{C}_{*-1}(P_1, \mathcal{F}_1)$ of complexes defined in (5), finishing the proof. $\square$

Let $(\mathbb{B}, \mathcal{F})$ be a coloured Boolean lattice of rank r and $\mathbb{B} = \mathbb{B}_0 \cup_f \mathbb{B}_1$ a decomposition of the form given in Example 9.

Corollary 23. *There is a long exact sequence*

$$\cdots \longrightarrow H_n(\mathbb{B}_1, \mathcal{F}_1) \xrightarrow{i_*} H_n(\mathbb{B}, \mathcal{F}) \xrightarrow{(\pi q)_*} H_{n-1}(\mathbb{B}_0, \mathcal{F}_0) \xrightarrow{\delta \pi_*^{-1}} H_{n-1}(\mathbb{B}_1, \mathcal{F}_1) \longrightarrow \cdots$$

4. The cube complex of a Boolean lattice and its homology

We now recall a construction, first due to Khovanov [Kho00], of a complex from a coloured Boolean lattice. It is central to the definition of the Khovanov homology of a link and is used in one of the recent combinatorial formulations of Heegaard–Floer knot homology. The reader should be aware that we are grading everything *homologically*, whereas in the applications cited above it is traditional to use cohomological conventions.

Let $\mathbb{B}$ be a Boolean lattice of rank r with ordered atoms $a_1, \ldots, a_r$, and colouring $\mathcal{F} : \mathbb{B} \to \mathcal{M}od_R$, and recall the unique expression (1) for an element of $\mathbb{B}$ as a join of the a_i (this replaces the conventions in earlier, non-lattice oriented, literature on Khovanov homology, where the elements of $\mathbb{B}$ were r-strings of 0's and 1's, and the atoms those r-strings containing a single 1). Write $1 := 1_{\mathbb{B}}$, the join of all the atoms.

If $x \prec y$, then let $\varepsilon(x \prec y) = (-1)^j$ where j is the number of atoms appearing before a_ℓ in the unique expression for y (see (1) and the comments following it). If $\mathbf{x} = x_1 \prec x_2 \prec \cdots \prec x_k$ is a saturated sequence in $\mathbb{B}$, let

$$\varepsilon_{\mathbf{x}} = \varepsilon(x_1 \cdots x_k) = \varepsilon(x_1 \prec x_2 \prec \cdots \prec x_k) := \prod \varepsilon(x_i \prec x_{i+1}).$$

If $1_0 = a_2 \vee \cdots \vee a_r$, then observe that $\varepsilon(1_0 \prec 1) = 1$.

Khovanov's *cube complex* $\mathcal{K}_*(\mathbb{B}, \mathcal{F})$ is then defined to have chain modules,

$$\mathcal{K}_k = \bigoplus_{\mathrm{rk}\, x = r-k} \mathcal{F}(x),$$

and differential $d_k : \mathcal{K}_k(\mathbb{B}, \mathcal{F}) \to \mathcal{K}_{k-1}(\mathbb{B}, \mathcal{F})$,

$$d_k(\lambda) = \sum \varepsilon(x \prec y) \mathcal{F}_x^y(\lambda),$$

where $\lambda \in \mathcal{F}(x)$ with $\mathrm{rk}\, x = r - k$, and the sum is over all y covering x. Thus, $d(\mathcal{K}_k) \subset \mathcal{K}_{k-1}$ with $d = \sum_{\mathrm{rk}\, x = r-k} \varepsilon(x \prec y) \mathcal{F}_x^y$. Observe that in degree zero the chains are just $\mathcal{F}(1)$, in degree r they are $\mathcal{F}(0)$, and $\mathcal{K}_k = 0$ outside of the range $0 \leqslant k \leqslant r$. To see that d is a differential, observe that if $x \prec z \prec y$ in $\mathbb{B}$, then there is a unique z' with $x \prec z' \prec y$, and that $\varepsilon(x \prec z \prec y) = -\varepsilon(x \prec z' \prec y)$, i.e. consecutive edges of the Hasse diagram for a Boolean lattice can always be completed to form a square in a unique way, and all squares anticommute. As d is a sum over such squares we get $d^2 = 0$. (See Fig. 3.)

Write $KH_*(\mathbb{B}, \mathcal{F}) = H_*(\mathcal{K}_*(\mathbb{B}, \mathcal{F}))$ for the homology of the cube complex. It should be noted that $KH(-)$ is not natural with respect to morphisms of coloured Boolean lattices in general. It is, however, natural with respect to morphisms (f, τ) for which f is a co-rank preserving injection.

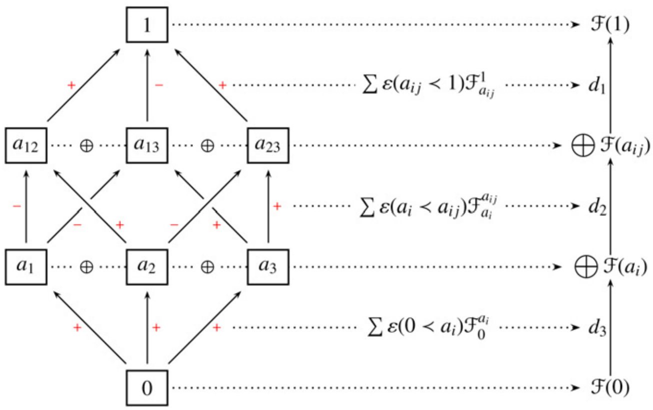


Fig. 3. The cube complex $\mathcal{K}_*(\mathbb{B}, \mathcal{F})$ for the Boolean lattice of rank 3 (after Bar-Natan [BN02]). The join $a_i \vee a_j$ has been abbreviated a_{ij}. The edges $x \prec y$ of the Hasse diagram for $\mathbb{B}$ have been labelled with the Khovanov signage $\varepsilon(x \prec y)$.

The Khovanov homology of an oriented link diagram is defined as (a normalised version of) the homology of the cube complex associated to the coloured Boolean lattice defined in Example 5. A small class of (graded) Frobenius algebras result in a homology theory that is invariant under Reidemeister moves of diagrams, thus giving a genuine invariant of links. The reader wishing to make this precise should be warned that here our homological grading conventions conflict with Khovanov's cohomological ones, and so care is needed (see Section 6).

Similarly, the combinatorial interpretation of Heegaard–Floer knot homology is defined as the homology of the cube complex associated to the coloured Boolean lattice of Example 6.

The decomposition $\mathbb{B} = \mathbb{B}_0 \cup_f \mathbb{B}_1$ of Example 9 yields a long exact sequence similar to that obtained for coloured poset homology described in the last section. As the $\mathbb{B}_i$ $(i = 0, 1)$ are Boolean of rank $r-1$, we may form the associated cube complexes $\mathcal{K}_*(\mathbb{B}_i, \mathcal{F}_i)$ where $\mathcal{F}_i$ is the restriction of $\mathcal{F}$ to $\mathbb{B}_i$. As with the complex $\mathcal{C}_*$ in Section 3, $\mathcal{K}_*(\mathbb{B}_1, \mathcal{F}_1)$ is a subcomplex of $\mathcal{K}_*(\mathbb{B}, \mathcal{F})$, but now the quotient is considerably simpler, for the map,

$$\sum_{\operatorname{rk} x = r-k} \lambda_x = \sum_{x \notin \mathbb{B}_0} \mu_x + \sum_{x \in \mathbb{B}_0} \nu_x \mapsto \sum_{x \in \mathbb{B}_0} \nu_x,$$

gives an isomorphism of complexes, $\mathcal{K}_*(\mathbb{B}, \mathcal{F})/\mathcal{K}_*(\mathbb{B}_1, \mathcal{F}_1) \to \mathcal{K}_{*-1}(\mathbb{B}_0, \mathcal{F}_0)$, and thus a short exact sequence,

$$0 \to \mathcal{K}_*(\mathbb{B}_1, \mathcal{F}_1) \to \mathcal{K}_*(\mathbb{B}, \mathcal{F}) \to \mathcal{K}_{*-1}(\mathbb{B}_0, \mathcal{F}_0) \to 0.$$

This sequence is well known, although the degree drop in our version happens in the quotient, rather than the subcomplex, as we are grading $\mathcal{K}_*$ homologically, rather than cohomologically. Finally, we have the induced long exact sequence in homology,

$$\cdots \longrightarrow KH_n(\mathbb{B}_1, \mathcal{F}_1) \longrightarrow KH_n(\mathbb{B}, \mathcal{F}) \longrightarrow KH_{n-1}(\mathbb{B}_0, \mathcal{F}_0) \longrightarrow KH_{n-1}(\mathbb{B}_1, \mathcal{F}_1) \longrightarrow \cdots$$

In Khovanov homology for links, if $(\mathbb{B}, \mathcal{F})$ is the coloured Boolean lattice of a diagram D (see Example 5) then $(\mathbb{B}_0, \mathcal{F}_0)$ and $(\mathbb{B}_1, \mathcal{F}_1)$ can be interpreted as the coloured lattices associated to diagrams D_0 and D_1 obtained from D by resolving a fixed crossing in D to a 0- and 1-smoothing respectively. In this case the above long exact sequence is a homological incarnation of the kind of skein relation found in the definition of certain knot polynomials.

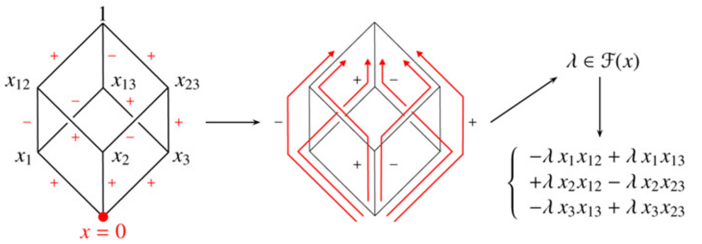


Fig. 4. The inclusion chain map $\phi : \mathcal{K}_*(\mathbb{B}, \mathcal{F}) \to \mathcal{C}_*(\mathbb{B}, \mathcal{F})$: the Boolean lattice of rank 3 (left) is marked with the Khovanov signage $\varepsilon(x \prec y)$; the saturated sequences $\mathbf{x}$ starting at x ($=0$ in this example) and finishing at 1 are marked (middle) with the resulting $\varepsilon_{\mathbf{x}}$, and the image (right) of $\lambda \in \mathcal{F}(x)$.

5. A quasi-isomorphism

We now have two chain complexes, and their homologies, associated to a coloured Boolean lattice: the coloured poset homology $H_*(\mathbb{B}, \mathcal{F})$ of the complex $\mathcal{C}_*(\mathbb{B}, \mathcal{F})$ from Section 2, and the homology $KH_*(\mathbb{B}, \mathcal{F})$ of the cube complex defined in Section 4. In this section we describe a chain map ϕ from the cube complex to $\mathcal{C}_*(\mathbb{B}, \mathcal{F})$, and show that it turns out to be a quasi-isomorphism. The main result is the following, whose proof appears at the end of the section.

Theorem 24. *Let $(\mathbb{B}, \mathcal{F})$ be a coloured Boolean lattice. Then $\phi : \mathcal{K}_*(\mathbb{B}, \mathcal{F}) \to \mathcal{C}_*(\mathbb{B}, \mathcal{F})$ defined below is a quasi-isomorphism, yielding isomorphisms,*

$$KH_n(\mathbb{B}, \mathcal{F}) \xrightarrow{\cong} H_n(\mathbb{B}, \mathcal{F}).$$

We now define the map ϕ. Let $\lambda \in \mathcal{F}(x)$ for $x \in \mathbb{B}$, and $\mathbf{x} = x_1 \prec \cdots \prec x_k$ a saturated sequence in $\mathbb{B}$ from x to 1, i.e. with $x_1 = x$ and $x_k = 1$, and let $\mathbf{x}^\circ = x_1 \prec \cdots \prec x_{k-1}$. Recalling the definition of $\varepsilon_{\mathbf{x}} \in \{\pm 1\}$ from Section 4, set $\phi : \mathcal{K}_n(\mathbb{B}, \mathcal{F}) \to \mathcal{C}_n(\mathbb{B}, \mathcal{F})$ to be

$$\phi(\lambda) = \sum_{\mathbf{x}} \varepsilon_{\mathbf{x}} \lambda \mathbf{x}^\circ,$$

the sum over all saturated sequences $\mathbf{x} \in \mathbb{B}$ from x to 1. (See Fig. 4.)

Proposition 25. *$\phi : \mathcal{K}_n(\mathbb{B}, \mathcal{F}) \to \mathcal{C}_n(\mathbb{B}, \mathcal{F})$ is a chain map.*

Proof. Is accomplished by a brute force comparison of the maps $\phi d_{\mathcal{K}}$ and $d_{\mathcal{C}}\phi$. Let $\lambda \in \mathcal{F}(x) \subset \mathcal{K}_n$, and $x_1, \ldots, x_{r-n}$ be the $r - n$ elements of $\mathbb{B}$ covering x. Then,

$$\lambda \overset{d_{\mathcal{K}}}{\longmapsto} \sum_{j=1}^{r-n} \varepsilon(xx_j) \mathcal{F}_x^{x_j}(\lambda) \overset{\phi}{\longmapsto} \sum_{j=1}^{r-n} \varepsilon(xx_j) \sum_{\mathbf{x}} \varepsilon(x_j \cdots x_{j_i} \prec 1) \mathcal{F}_x^{x_j}(\lambda) x_j \cdots x_{j_i},$$

with the second summation over the saturated sequences $\mathbf{x}$ from x_j to 1. On the other hand,

$$\lambda \overset{\phi}{\longmapsto} \sum_{j=1}^{r-n} \varepsilon(xx_j) \sum_{\mathbf{x}} \varepsilon(x_j \cdots x_{j_i} \prec 1) \lambda x x_j \cdots x_{j_i},$$

with again the second sum over the saturated chains $\mathbf{x}$ from x_j to 1. In the image under $d_{\mathcal{C}}$, each of the $r-n$ terms contributes a term of the form $\varepsilon(xx_j)\sum_{\mathbf{x}}\varepsilon(x_j\cdots x_{j_i}\prec 1)\mathcal{F}_x^{x_j}(\lambda)x_j\cdots x_{j_i}$, obtained by dropping the x from the chain $xx_j\cdots x_{j_i}$. All the other terms have the form

$$\varepsilon(xx_j)\varepsilon(x_j\cdots x_{j_i}\prec 1)(-1)^k\lambda xx_j\cdots\widehat{x}_k\cdots x_{j_i}, \tag{6}$$

for $j\leqslant k\leqslant j_i$, and where $\varepsilon(xx_j)\varepsilon(x_j\cdots x_{j_i}\prec 1)=\varepsilon(xx_j\cdots x_{j_i}\prec 1)$. The proof is thus completed by showing that all these terms cancel. As already observed, for any chain $x_{k-1}\prec x_k\prec x_{k+1}$ in $\mathbb{B}$ there is a unique $y_k\neq x_k$ with $x_{k-1}\prec y_k\prec x_{k+1}$, and $\varepsilon(x_{k-1}x_kx_{k+1})=-\varepsilon(x_{k-1}y_kx_{k+1})$. Thus, there is a matching term to (6), indexed by $xx_j\cdots\widehat{y}_k\cdots x_{j_i}$, and otherwise identical in all respects except for having opposite sign. This completes the proof. $\square$

We now bring in the decomposition $\mathbb{B}=\mathbb{B}_0\cup_f\mathbb{B}_1$ of the Boolean lattice of Example 9 for $\ell=1$. Notice that if $x\in\mathbb{B}_1$ and $\mathbf{x}$ is a sequence (saturated or not) starting at x, then $\mathbf{x}$ is completely contained in the sublattice $\mathbb{B}_1$. Thus in particular, when $\lambda\in\mathcal{F}(x)$ we have $\phi(\lambda)$ is in the subcomplex $\mathcal{C}_*(\mathbb{B}_1,\mathcal{F}_1)\subset\mathcal{C}_*(\mathbb{B},\mathcal{F})$, and so $\phi\mathcal{K}_*(\mathbb{B}_1,\mathcal{F}_1)\subset\mathcal{C}_*(\mathbb{B}_1,\mathcal{F}_1)$. We therefore have an induced map of complexes,

$$\phi':\mathcal{K}_{*-1}(\mathbb{B}_0,\mathcal{F}_0)=\frac{\mathcal{K}_*(\mathbb{B},\mathcal{F})}{\mathcal{K}_*(\mathbb{B}_1,\mathcal{F}_1)}\to\frac{\mathcal{C}_*(\mathbb{B},\mathcal{F})}{\mathcal{C}_*(\mathbb{B}_1,\mathcal{F}_1)}=Q_*.$$

Lemma 26. *Let $\pi:Q_*\to\mathcal{C}_{*-1}(\mathbb{B}_0,\mathcal{F}_0)$ be the map defined in Section 3 by Eq. (5) and ϕ,ϕ' as above. Then the following diagram of chain maps commutes.*

$$\begin{tikzcd}
\mathcal{K}_{*-1}(\mathbb{B}_0,\mathcal{F}_0) \arrow[r, "\phi'"] \arrow[dr, "\phi"'] & Q_* \arrow[d, "\pi"] \\
 & \mathcal{C}_{*-1}(\mathbb{B}_0,\mathcal{F}_0)
\end{tikzcd}$$

Note that the ϕ that appears in the diagram is the ϕ associated to the sublattice $\mathbb{B}_0$ (not $\mathbb{B}$).

Proof. Let $x\in\mathbb{B}_0$ and S be the set of all saturated sequences $\mathbf{x}=x_1\cdots x_jy_1\cdots y_{n-j}1=x_1\prec\cdots\prec x_j\prec y_1\prec\cdots\prec y_{n-j}\prec 1$ in $\mathbb{B}$ with the $x_i\in\mathbb{B}_0$, $y_i\in\mathbb{B}_1$ and $x_1=x$. Let $S'\subset S$ consist of those saturated sequences of the form $x_1\cdots x_n1$, where the $x_i\in\mathbb{B}_0$, $x_1=x$ and $x_n=1_0$, the unique maximal element of $\mathbb{B}_0$. Then, for $\lambda\in\mathcal{F}(x)$ we have

$$\lambda\overset{\phi'}{\longmapsto}\sum_{\mathbf{x}\in S}\varepsilon_{\mathbf{x}}\lambda\mathbf{x}^\circ\overset{\pi}{\longmapsto}\sum_{\mathbf{x}\in S'}\varepsilon_{\mathbf{x}}\lambda x_1\cdots x_{n-1}.$$

Now the $\varepsilon_{\mathbf{x}}$ that appears on the right-hand side above satisfies $\varepsilon_{\mathbf{x}}=\varepsilon(x_1\prec\cdots\prec x_{n-1}\prec 1_0\prec 1)=\varepsilon(x_1\prec\cdots\prec 1_0)\varepsilon(1_0\prec 1)$, which in turn is just $\varepsilon(x_1\prec\cdots\prec 1_0)$, as $\varepsilon(1_0\prec 1)=1$. $\square$

In particular we have a commuting diagram in homology: $\phi_*=\pi_*\phi'_*$. We now have everything we need for the proof of the Main Theorem:

Proof of the Main Theorem. The short exact sequences in Sections 3 and 4 can be assembled into a diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_*(\mathbb{B}_1,\mathcal{F}_1) \arrow[r] \arrow[d, "\phi"'] & \mathcal{K}_*(\mathbb{B},\mathcal{F}) \arrow[r] \arrow[d, "\phi"'] & \mathcal{K}_{*-1}(\mathbb{B}_0,\mathcal{F}_0) \arrow[r] \arrow[d, "\phi'"] & 0 \\
0 \arrow[r] & \mathcal{C}_*(\mathbb{B}_1,\mathcal{F}_1) \arrow[r] & \mathcal{C}_*(\mathbb{B},\mathcal{F}) \arrow[r] & Q_* \arrow[r] & 0
\end{tikzcd}$$

where by definition, ϕ' is the map making the right-hand square commute, while it is easy to check that the left-hand square commutes. By the functoriality of the long exact sequence in homology, we have a commutative diagram containing the following portion

$$\begin{tikzcd}
KH_n(\mathbb{B}_0,\mathcal{F}_0) \arrow[r, "\delta"] \arrow[d, "\phi'_*"] & KH_n(\mathbb{B}_1,\mathcal{F}_1) \arrow[r] \arrow[d, "\phi_*"] & KH_n(\mathbb{B},\mathcal{F}) \arrow[r] \arrow[d, "\phi_*"] & KH_{n-1}(\mathbb{B}_0,\mathcal{F}_0) \arrow[r, "\delta"] \arrow[d, "\phi'_*"] & KH_{n-1}(\mathbb{B}_1,\mathcal{F}_1) \arrow[d, "\phi_*"] \\
H_{n+1}(Q_*) \arrow[r, "\delta"] & H_n(\mathbb{B}_1,\mathcal{F}_1) \arrow[r] & H_n(\mathbb{B},c) \arrow[r] & H_n(Q_*) \arrow[r, "\delta"] & H_{n-1}(\mathbb{B}_1,\mathcal{F}_1)
\end{tikzcd}$$

where the rows are exact. The proof then proceeds by induction on the rank, noting that the result is obviously true for Boolean lattices of rank 1. If $\mathbb{B}$ is rank $r+1$ then both $\mathbb{B}_0$ and $\mathbb{B}_1$ are rank r, so assuming the result for rank r gives that the second and fifth vertical maps in the above diagram are isomorphisms. Furthermore, the first and fourth maps are also isomorphisms: Lemma 26 gives that the $\phi'_* = \pi_*^{-1}\phi_*$, where ϕ_* is again an isomorphism because $\mathbb{B}_0$ has rank r, and π_* is an isomorphism by Proposition 20. By the 5-lemma, the middle map is thus an isomorphism too. □

The Main Theorem can be strengthened somewhat: if P is a poset, then call $\mathcal{F}: P \to \mathcal{M}\mathrm{od}_R$ a *colouring by projectives* if $\mathcal{F}(x)$ is a projective module for all $x \in P$. As the direct sum of projectives is projective, and a quasi-isomorphism between bounded below chain complexes of projectives is a homotopy equivalence (see, e.g. [Wei94, §10.4]), we get that,

Corollary 27. *If $\mathcal{F}$ is a colouring by projectives then $\phi: \mathcal{K}_n(\mathbb{B},\mathcal{F}) \to \mathcal{C}_n(\mathbb{B},\mathcal{F})$ is a homotopy equivalence.*

In particular, as vector spaces are projective we have,

Corollary 28. *If the ground ring R of the colouring $\mathcal{F}: \mathbb{B} \to \mathcal{M}\mathrm{od}_R$ is a field, then $\phi: \mathcal{K}_n(\mathbb{B},\mathcal{F}) \to \mathcal{C}_n(\mathbb{B},\mathcal{F})$ is a homotopy equivalence.*

6. Normalisation for link homology

For the motivating example, namely the Khovanov colouring of a Boolean lattice associated to a link diagram, the modules are in fact graded, and in order to obtain an invariant some shifts are required. We record these shifts here in order to minimise the potential confusion arising from our grading conventions.

Let V be the graded Frobenius algebra used in the construction of Khovanov homology and let $(\mathbb{B},\mathcal{F})$ be the Boolean lattice associated to a given link diagram coloured with the Khovanov colouring of Example 5. The grading on V induces an internal grading on the associated complex. Using the convention that $(W_{*,*}[a,b])_{i,j} = W_{i-a,j-b}$, the shifted complex we wish to consider is $\tilde{\mathcal{S}}_{*,*}(\mathbb{B},\mathcal{F}) = \mathcal{S}_{*,*}(\mathbb{B},\mathcal{F})[-n_+, n_+ - 2n_-]$, i.e.

$$\tilde{\mathcal{S}}_{i,j}(\mathbb{B},\mathcal{F}) = \mathcal{S}_{i+n_+,\,j-n_++2n_-}(\mathbb{B},\mathcal{F})$$

where n_+ and n_- are the number of positive and negative crossings of the (oriented) diagram. The homology $\tilde{H}_{*,*}(\mathbb{B},\mathcal{F})$ is then a bigraded link invariant. To compare with the more usual grading in Khovanov homology we have $KH^{i,j} \cong \tilde{H}_{-i,j}$.

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